

TO

Exact Summation of the
Chapman - Enskog Expansion

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References

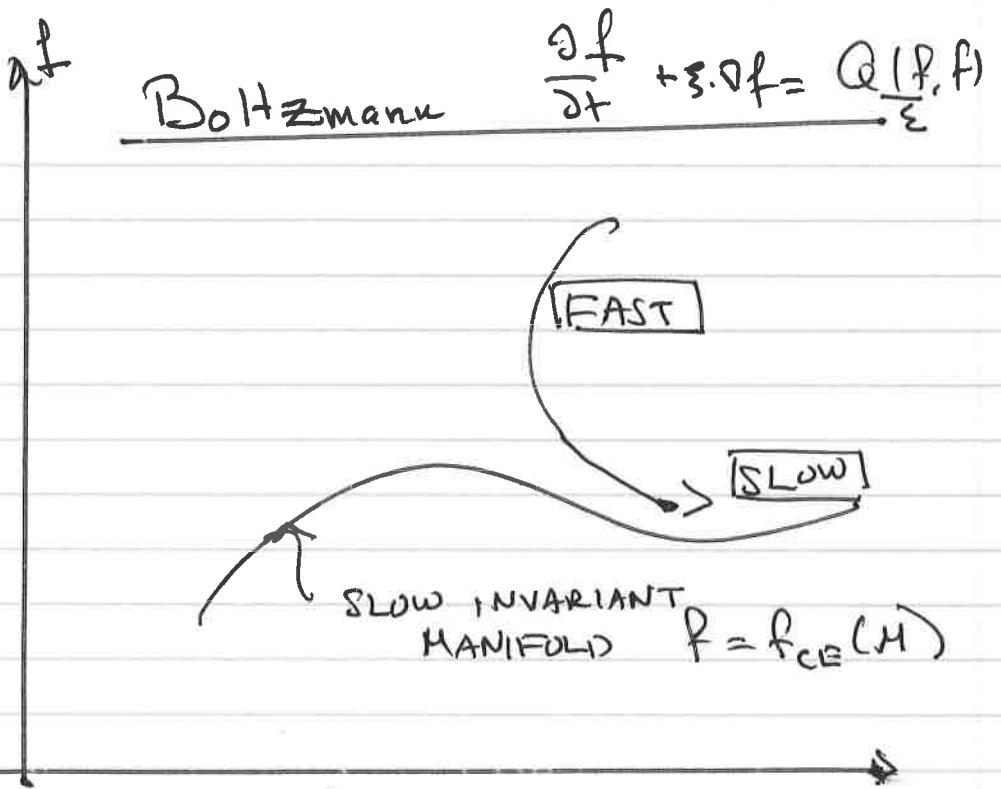
A. Gorban and I. Karlin

Sov. Physics JETP 73 (1991)

Phys. Rev. Letters 77 (1996)

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M = hydrodynamic fields

$$f_{CE}(M) = f^{(0)}(M) + \varepsilon f^{(1)}(M) + \varepsilon^2 f^{(2)}(M) + \dots$$

[local
equil
(Euler)]

NSF

Burnett

Grad's B moments

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Linearize about $\rho_0, T_0, \underline{u} = 0$.

Viscosity $\mu(T) = \eta(\tau)T$

$\eta = \text{const}$ for Maxwell molecules.
 $\sim \sqrt{T}$ for hard spheres.

Introduce dimensionless variables:

$$\underline{u} = \frac{\delta \underline{u}}{\sqrt{R_B T_0 / m}}, \quad \underline{s} = \frac{\delta \underline{s}}{\delta \underline{s}_0}, \quad \underline{T} = \frac{\delta \underline{T}}{T_0}$$

$$\underline{x} = \frac{\rho_0 \underline{x}'}{\eta(T_0) \sqrt{R_B T_0 / m}}, \quad t = \frac{\rho_0}{\eta(T_0)} + 1$$

$$R_B \rightarrow 1, m \rightarrow 1$$

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Grad's 13 moments : Linearized

$$\partial_t f = - \nabla \cdot \underline{u}$$

$$\partial_t \underline{u} = \underbrace{-\nabla f - \nabla T}_{-\nabla p} - \nabla \cdot \underline{\sigma}$$

$$\partial_t T = -\frac{2}{3} (\nabla \cdot \underline{u} + \nabla \cdot \underline{q})$$

$$\partial_t \underline{\sigma} = -((\nabla \underline{u}) + (\nabla \underline{u})^T - \frac{2}{3} \underline{\underline{I}} \nabla \cdot \underline{u})$$

$$-\frac{2}{5} ((\nabla \underline{q}) + (\nabla \underline{q})^T - \frac{2}{3} \underline{\underline{I}} \nabla \cdot \underline{q})$$

$$\partial_t \underline{q} = -\frac{5}{3} \nabla T - \nabla \cdot \underline{\sigma} - \frac{2}{3} \underline{\underline{q}}$$

$$P = f + T$$

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Linearized 10 moments : one space dim

$$P = p + T \quad , \quad \left[\begin{array}{l} \partial_t f = -ux \\ \partial_t T = -\frac{2}{3}ux \end{array} \right] \text{ Add}$$

$$\boxed{\begin{aligned} \partial_t P &= -\frac{5}{3}ux \\ \partial_t u &= -Px - Gx \\ \partial_t \sigma &= -\frac{4}{5}ux - \sigma \end{aligned}}$$

T6

$$t = \bar{t}/\epsilon, \quad x = \bar{x}/\epsilon \Rightarrow$$

$$\partial_t p = -\frac{5}{3} u_x$$

$$\partial_t u = -p_x - \sigma_x$$

$$\sigma_x = -\frac{4}{3} u_x - \frac{\sigma}{\epsilon}$$

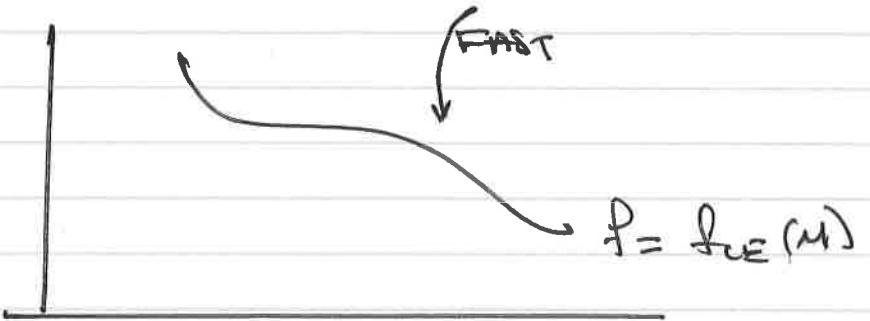
LG

ϵ Knudsen number

$$\sigma = e^{-t/\epsilon} \sigma(x, 0) + \int_0^L e^{(s-t)/\epsilon} \left(-\frac{4}{3} u_x(s, x) \right) ds$$

This gives visco-elastic, not hydrodynamics

Only gives fast part



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Chapman-Euskog

$$\sigma_{CE} = \epsilon \sigma^{(0)} + \epsilon^2 \sigma^{(1)} + \epsilon^3 \sigma^{(2)} + \epsilon^4 \sigma^{(3)} + \dots$$

 $\sigma^{(n)}$

depend only on current
values of p, u and
their space derivatives

$$P_t = -\frac{5}{3} u_x$$

$$u_t = -p_x - (\epsilon \sigma^{(0)} + \epsilon^2 \sigma^{(1)} + \epsilon^3 \sigma^{(2)} + \epsilon^4 \sigma^{(3)} + \dots)_x \\ (\epsilon \sigma_t^{(0)} + \epsilon^2 \sigma_t^{(1)} + \epsilon^3 \sigma_t^{(2)} + \epsilon^4 \sigma_t^{(3)} + \dots) =$$

$$\underline{\underline{-\frac{4}{3} u_x - (\sigma^{(0)} + \epsilon \sigma^{(1)} + \epsilon^2 \sigma^{(2)} + \epsilon^3 \sigma^{(3)} + \dots)}} =$$

$$\Rightarrow \boxed{\sigma^{(0)} = -\frac{4}{3} u_x}$$

$$\Rightarrow u_t = -p_x - (-\epsilon \frac{4}{3} u_x + \epsilon^2 \sigma^{(1)} + \epsilon^3 \sigma^{(2)} + \epsilon^4 \sigma^{(3)} + \dots)_x$$

$$\left(-\frac{4}{3} \epsilon u_{xt} + \epsilon^2 \sigma_t^{(0)} + \epsilon^3 \sigma_t^{(1)} + \epsilon^4 \sigma_t^{(2)} + \dots \right) = \\ \downarrow \quad \rightarrow (\epsilon \sigma^{(0)} + \epsilon^2 \sigma^{(1)} + \epsilon^3 \sigma^{(2)} + \dots)$$

Eliminate time derivative

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$$-\frac{4}{3} \epsilon (-P_{xx} + \underbrace{\epsilon \frac{4}{3} u_{xxx} + \epsilon^2 \sigma_{xx}^{(1)} + \dots}_{\sim}) \\ + \underbrace{\epsilon^2 \sigma_{xx}^{(2)}}_{\sim} + \epsilon^3 \sigma_{xx}^{(3)} + \epsilon^4 \sigma_{xx}^{(4)} + \dots =$$

$$- \left(\underbrace{\epsilon \sigma^{(1)}}_{\sim} + \underbrace{\epsilon^2 \sigma^{(2)}}_{\sim} + \epsilon^3 \sigma^{(3)} + \dots \right)$$

$$\Rightarrow \boxed{\sigma^{(1)} = -\frac{4}{3} P_{xx}}$$

$$-\frac{4}{3} \left(\frac{4}{3} u_{xxx} + \sigma_{xx}^{(1)} \right) = -\sigma^{(2)}$$

$$-\frac{4}{3} \left(\frac{4}{3} u_{xxx} - \frac{4}{3} P_{xxx} \right) = -\sigma^{(2)}$$

$$-\frac{4}{3} \left(\frac{4}{3} u_{xxx} + P_{xxx} \right) = -\sigma^{(2)}$$

$$-\frac{4}{3} \left(\frac{4}{3} u_{xxx} - \frac{5}{3} u_{xxx} \right) = -\sigma^{(2)}$$

$$-\left(-\frac{4}{3} \right) \left(\frac{1}{3} \right) u_{xxx} = \sigma^{(2)}$$

$$\boxed{\sigma^{(2)} = -\frac{4}{3} \left(\frac{1}{3} \right) u_{xxx}}$$

(9)

$$\tilde{\sigma}_{CE} = -\frac{4}{3} (\epsilon u_x + \epsilon^2 p_{xx} + \frac{\epsilon^3}{3} u_{xxx} + \dots)$$

Dispersion relation : spectrum via Fourier space

NS: $\omega_{\pm} = -\frac{2}{3} k^2 \pm i|k| \sqrt{4k^2 - 15} : O(\epsilon)$

Burnett: $\omega_{\pm} = -\frac{2}{3} k^2 \pm i|k| \sqrt{8k^2 + 15} : O(\epsilon^2)$

Super Burnett : $O(\epsilon^3)$

$$\omega_{\pm} = \frac{2}{9} k^2 (k^2 - 3) \pm \frac{1}{9} i |k| \sqrt{9k^6 - 24k^4 - 72k^2 - 135}$$

Hence,

NS, Burnett $\operatorname{Re} \omega_{\pm}(k) \leq 0$

Super Burnett $\operatorname{Re} \omega_I(k) > 0$

for $|k| > \sqrt{3}$

"Babylev instability"

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Q. What is the mistake here?

A. The mistake is not where to
truncates C - E , the mistake
is truncation itself.

References

P. Rosenau , Phys Rev. A 40 (1989) ,

D. Rosenau , Phys. Rev. A 39 (1989) .

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NOTICE IN C-E:

$$\sigma^{(2n)} = a_n \partial_x^{2n+1} u, \quad (1)$$

$$\sigma^{(2n+1)} = b_n \partial_x^{2(n+1)} p \quad (2)$$

$$v_{CE} = \sum_{n=0}^{\infty} \epsilon^{2n+1} a_n \partial_x^{2n+1} u$$

$$+ \sum_{n=0}^{\infty} \epsilon^{2n+2} b_n \partial_x^{2(n+1)} p$$

$$\frac{x}{\epsilon} \rightarrow x \quad \frac{t}{\epsilon} \rightarrow t$$

$$v_{CE} = \sum_{n=0}^{\infty} a_n \partial_x^{2n+1} u + \sum_{n=0}^{\infty} b_n \partial_x^{2(n+1)} p.$$

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Recursion relation for C-E:

$$\sigma^{(n)} = - \sum_{m=0}^{n-1} \underline{\partial_x}^{(m)} \sigma^{(n-1-m)} | (3)$$

$$\underline{\partial_x}^{(m)} \underline{\partial_x}^l u = \text{defn. } \left\{ \begin{array}{l} - \underline{\partial_x}^{l+1} p, \quad m=0 \\ - \underline{\partial_x}^{l+1} \sigma^{(m-1)}, \quad m \geq 1 \end{array} \right.$$

$$\underline{\partial_x}^{(m)} \underline{\partial_x}^l p = \text{defn. } \left\{ \begin{array}{l} - \frac{5}{3} \underline{\partial_x}^{l+1} u, \quad m=0 \\ 0, \quad m \geq 1 \end{array} \right.$$

$$\sigma^{(0)} = - \frac{4}{3} ux$$

(13)

Now substitute (1), (2) into (3)

$$b_n \partial_x^{2(n+1)} p = \sum_{m=1}^n a_{n-m} \partial_x^{2(n-m+1)} b_{m-1} \partial_x^{2m} p + a_n \partial_x^{2(n+1)} p$$

$$a_n \partial_x^{2n+1} u = \frac{5}{3} b_{n-1} \partial_x^{2n+1} u + \sum_{m=0}^{n-1} a_{n-m-1} \partial_x^{2(n-m)} a_m \partial_x^{(2m+1)} u$$

$$b_{n+1} = a_{n+1} + \sum_{m=0}^n a_{n-m} b_m , \quad (4)$$

$$a_{n+1} = \frac{5}{3} b_n + \sum_{m=0}^n a_m a_{n-m}$$

Easier to work in freq domain

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$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx. \quad F.T.$$

$$\hat{f}_{CE} = \sum_{n=0}^{\infty} a_n (-ik)^{2n+1} \hat{u}$$

$$+ \sum_{n=0}^{\infty} b_n (-ik)^{2(n+1)} \hat{p}$$

$$[(-ik)^{2n+1} = (-ik) ((-ik)^2)^n = -ik (-k^2)^n]$$

$$\hat{f}_{CE} = \sum_{n=0}^{\infty} -ik a_n (-k^2)^n \hat{u}$$
$$+ \sum_{n=0}^{\infty} -k^2 b_n (-k^2)^n \hat{p}$$

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$$\begin{aligned} \nabla_{\text{ext}} &= -ie A(r^2) \hat{u} \\ &\quad - R^2 B(r^2) \hat{p} \end{aligned}$$

(5)

$$A(r^2) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} a_n (-r^2)^n$$

$$B(r^2) \stackrel{\text{def.}}{=} \sum_{n=0}^{\infty} b_n (-r^2)^n$$

$$\left(a_{n+1} = \frac{5}{3} b_n + \sum_{m=0}^n a_m a_{n-m} \right) (-k^2)^{n+1}$$

$$\left(b_{n+1} = a_{n+1} + \sum_{m=0}^{n+1} a_{n-m} b_m \right) (-k^2)^{n+1}$$

Sum and change order of summation \Rightarrow

$$A - a_0 = -R^2 \left\{ \frac{5}{3} B + A^2 \right\}$$

(6)

$$B - b_0 = A - a_0 - R^2 AB$$

$$a_0 = b_0 = -\frac{4}{3}$$

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Solve for A: $A = \frac{B}{1 - r^2 B}$
 in 2nd eqn

Substitute into 1st eqn:

Define $C(r^2) \stackrel{\text{def}}{=} r^2 B(r^2) \Rightarrow$

$$-\frac{5}{3} (1 - C(r^2))^2 (C(r^2) + \frac{4}{5}) = \frac{C(r^2)}{r^2} \quad (M)$$

cubic

$$C = y + \frac{2}{5}$$

$$y^3 + \frac{3}{5} \left(-\frac{7}{5} + \frac{1}{r^2} \right) y + \frac{6}{5} \left(3 + \frac{1}{5r^2} \right) = 0$$

$$y^3 + ay + b = 0$$

$$\frac{b^2}{4} + \frac{a^3}{27} > 0 \Rightarrow \text{one real root.}$$

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But

$$F(C) = -\frac{5}{3}(-C)^2(C + \frac{4}{5}) - \frac{C}{R^2}$$

$$F(C) > 0, \quad C < -\frac{4}{5}$$

$$F(C) < 0, \quad C > 1$$

$$F(0) = -\frac{4}{3}$$

So SIGN CHANGE MUST OCCUR

$$\text{on } (-\frac{4}{5}, 0)$$

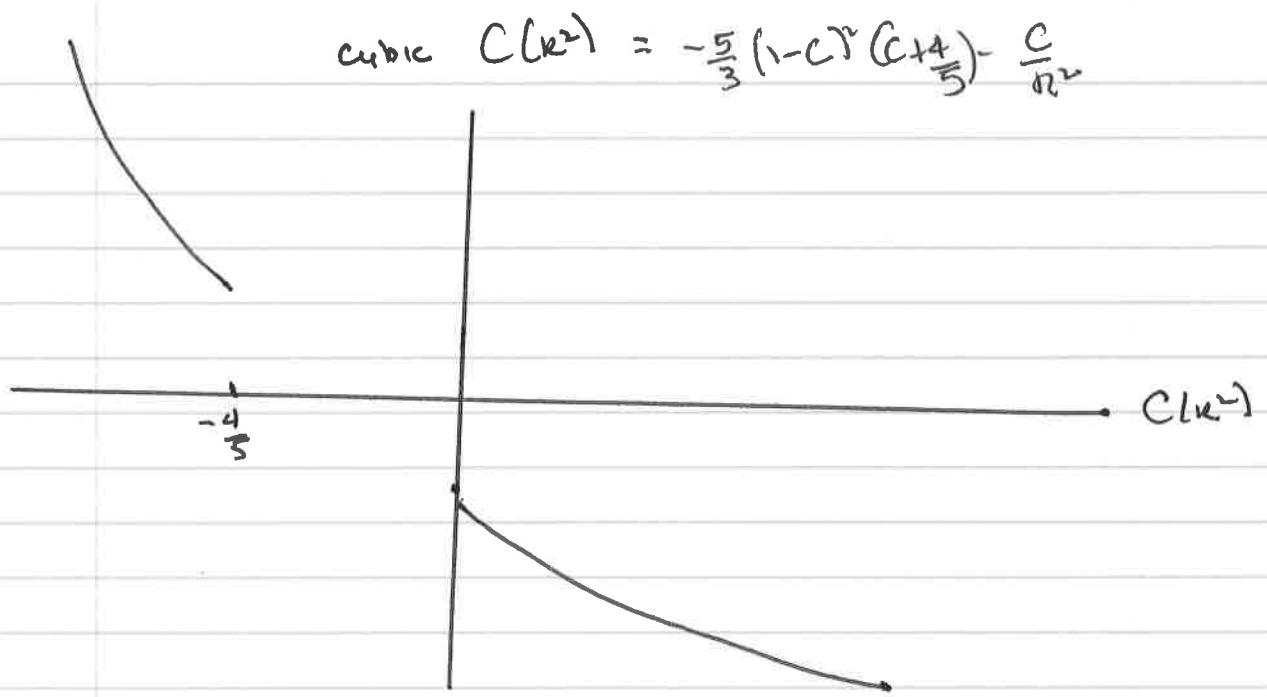
ALSO

$$\frac{dC}{dR^2} = \left\{ -\frac{10}{3}(-C)(C + \frac{4}{5}) \right. \\ \left. - \frac{5}{3}(-C)^2 - \frac{1}{R^4} \right\} = -\frac{C}{R^4}$$

$$\text{sign } \frac{dC}{dR^2} = \text{sign } \frac{C}{R^4} = \text{sign } C$$

$$\text{so } \frac{dC}{dR^2} < 0 -$$

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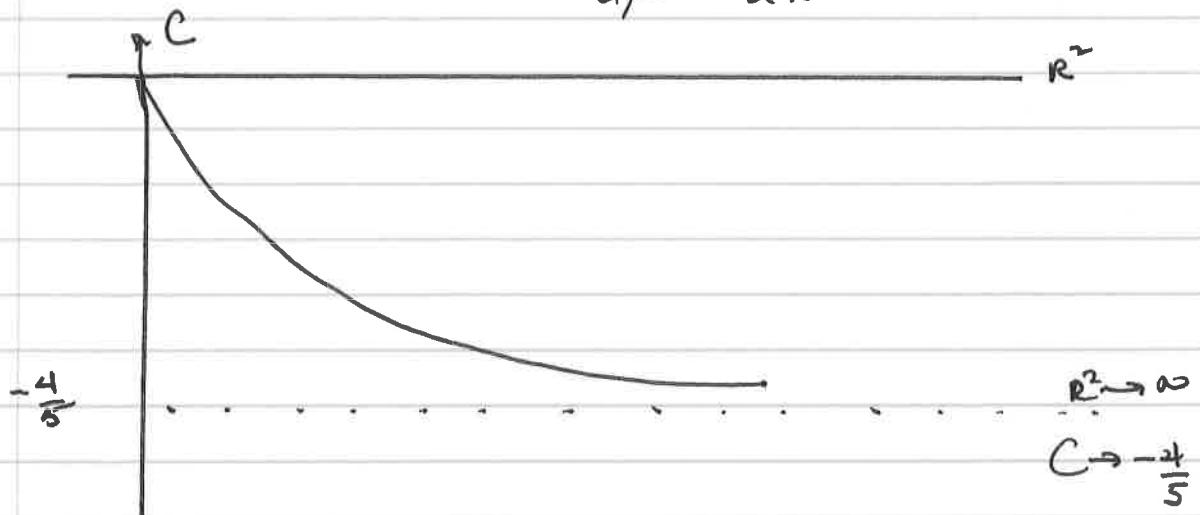


HENCE REAL ROOT OF CUBIC

IS BETWEEN $C = 0, C = -\frac{4}{5}$

AND MONOTONE IN $\mu = k^2$

IN THE INTERVAL $\frac{dC}{d\mu} = \frac{dC}{dk^2} < 0$.



What is α_{CE} ?

$$\alpha_{CE} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{\sigma}_{CE} e^{ikx} dk$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (-i\omega A(\omega) \hat{u} - \omega^2 B(\omega) \hat{p}) e^{ikx} dk$$

where

$$A = \frac{B}{1 - \omega^2 B} = \left(\frac{Br^2}{1 - r^2 B} \right) \frac{1}{r^2}$$

$$A = \left(\frac{C}{1 - C} \right) \frac{1}{r^2}$$

$$C = Br^2$$

$$B = \frac{C}{r^2}$$

EXTRA STRESS IS FOURIER INTEGRAL OPERATOR

EQNS OF HYDRO DYNAMICS

$$\dot{P}_x = -\frac{5}{3} u_x$$

$$u_x = -P_x - \tilde{\sigma}_x$$

$$\hat{P}_x = -\frac{5}{3} (-ik) \hat{u}$$

$$\hat{u}_x = -(-ik) \hat{P} - \hat{\sigma} (-ik)$$

$$\hat{P}_x = \frac{5}{3} ik \hat{u}$$

$$\hat{u}_x = i \kappa \hat{P}$$

$$ik \left(-ik A(u^z) \hat{u} - k^2 B(u^z) \hat{P} \right)$$

$$\hat{P}(t, r) = e^{wt} P(r), \quad \hat{u}(t, r) = e^{wt} U(r)$$

$$\begin{bmatrix} -w & +\frac{5}{3} ik \\ ik - ik^3 B & k^2 A - w \end{bmatrix} \begin{bmatrix} P \\ U \end{bmatrix} = \begin{bmatrix} 0 \\ \omega \end{bmatrix}$$

→

$$\det = 0 \Rightarrow$$

$$\omega^2 - \omega k^2 A + \frac{5}{3} (r^2 - r^4 B) = 0$$

$$\omega^2 - \omega \left(\frac{C}{1-C} \right) + \frac{5}{3} k^2 (1-C) = 0$$

$$\omega = \frac{1}{2} \left(\frac{C}{1-C} \right) \pm \omega_1 \left(\left(\frac{C}{1-C} \right)^2 \frac{1}{k^2} - \frac{20}{3} (1-C) \right)^{1/2}$$

But recall C is root of cubic

$$-\frac{5}{3} (C-1)^2 (C + \frac{4}{5}) = \frac{C}{k^2}$$

$$\Rightarrow -\frac{5}{3} C \left(C + \frac{4}{5} \right) = \frac{C^2}{(1-C)^2} \frac{1}{k^2}$$

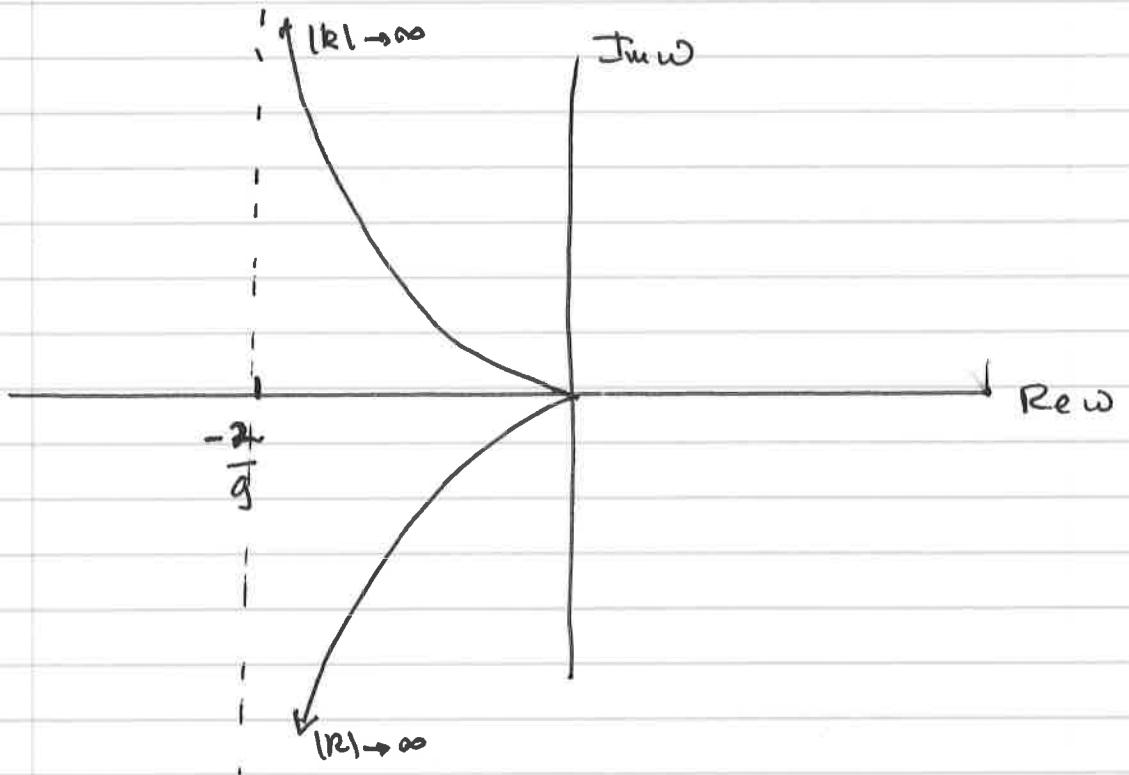
$$\omega = \frac{1}{2} \left(\frac{C}{1-C} \right) \pm \omega_1 \left(\frac{5C^2 - 16C + 20}{3} \right)^{1/2}$$

$$5C^2 - 16C + 20 > 0$$

(22)

$$\operatorname{Re}\omega \Rightarrow \frac{1}{2} \left(\frac{-\frac{4}{5}}{1 + \frac{4}{5}} \right) = \frac{1}{2} \left(\frac{-\frac{4}{5}}{\frac{9}{5}} \right) = -\frac{2}{9}$$

as $|\omega| \rightarrow \infty$



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New material

The entropy equality.

$$P_t = -\frac{5}{3} u_x, \quad u_t = -p_x - \sigma_x$$

$$\hat{P}_t = \frac{5}{3} i k \hat{u}, \quad \hat{u}_t = i k (\hat{p} + \hat{\sigma}) \quad F.T.$$

$$\hat{P}_t = \frac{5}{3} i k \hat{u}$$

$$\begin{aligned} \hat{u}_t = & i k \hat{p} + k^2 A(k) \hat{u} \\ & + i k (-k^2 B(k) \hat{p}) \end{aligned}$$

$$\left(\frac{5}{3} \hat{P}_t = i k \hat{u} \right) \hat{p}$$

$$\left(\hat{u}_t = i k \hat{p} + k^2 A(k) \hat{u} \right. \\ \left. + i k (-k^2 B(k) \hat{p}) \right) \hat{u}$$

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$$\frac{1}{2} \partial_t \left(\frac{3}{5} |\vec{p}|^2 + 1G^2 \right) - \nu k (\vec{p} \hat{u} + \vec{p} \bar{\hat{u}})$$

$$= r^2 A(r^2) |\hat{u}|^2$$

$$+ \nu k \bar{\hat{u}} (-r^2 B(r^2) \vec{p})$$

$$\frac{3}{5} \vec{p}_t = i \nu k \hat{u}$$

$$\frac{3}{5} \vec{p}_t = -i \nu k \bar{\hat{u}}$$

$$\nu k \bar{\hat{u}} (-r^2 B(r^2) \vec{p}) =$$

$$-\frac{3}{5} \vec{p}_t (-r^2 B(r^2) \vec{p}) =$$

$$\frac{1}{2} \frac{3}{5} r^2 B(r^2) \partial_t |\vec{p}|^2$$

$$\frac{1}{2} \left[\partial_t \left(\frac{3}{5} |\vec{p}|^2 + 1G^2 - \frac{3}{5} r^2 B(r^2) |\vec{p}|^2 \right) \right]$$

$$-i \nu k (\vec{p} \hat{u} + \vec{p} \bar{\hat{u}}) = r^2 A(r^2) |\hat{u}|^2$$

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Parseval \Rightarrow

$$\begin{aligned} & \frac{1}{2} \partial_t \int_{-\infty}^{\infty} \frac{3}{5} |\hat{p}|^2 + |\hat{u}|^2 \, dr \\ & + \frac{1}{2} \partial_t \int_{-\infty}^{\infty} -\frac{3}{5} r^2 B(r^2) |\hat{p}|^2 \, dr \\ & \rightarrow \int_{-\infty}^{\infty} \overline{\frac{\partial \hat{p}}{\partial x}} \hat{u} + \overline{\frac{\partial \hat{u}}{\partial x}} \hat{p} \, dr \\ & = \int_{-\infty}^{\infty} r^2 A(r^2) |\hat{u}|^2 \, dr \end{aligned}$$

$$\begin{aligned} & \frac{1}{2} \partial_t \int_{-\infty}^{\infty} \frac{3}{5} |up|^2 + |ur|^2 \, dx \\ & + \frac{1}{2} \partial_t \int_{-\infty}^{\infty} -\frac{3}{5} r^2 B(r^2) |\hat{p}|^2 \, dr \\ & = \int_{-\infty}^{\infty} r^2 A(r^2) |\hat{u}|^2 \, dr \end{aligned}$$

$B < 0, \quad A < 0.$

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Recall $C(r^2) = r^2 B(r^2)$

$$r^2 A(r^2) = \frac{C(r^2)}{1 - C(r^2)}$$

"Navier Stokes" small r^2 approx.

$$C(r^2) \approx C_0 r^2 \quad C_0 < 0$$

$$\Rightarrow C_0 r^2 = r^2 B(r^2)$$

$$B = C_0.$$

$$r^2 A(r^2) = \frac{C_0 r^2}{1 - C_0 r^2}$$

$$= C_0 r^2$$

$$\frac{1}{2} \eta + \int_{-\infty}^{\infty} \frac{3}{5} |p|^2 + |u|^2 dx$$

$$+ \frac{1}{2} \frac{3}{5} \int_{-\infty}^{\infty} \|C_0\| \left| \frac{\partial p}{\partial x} \right|^2 dx$$

$$= - \int_{-\infty}^{\infty} \|C_0\| \left| \frac{\partial u}{\partial x} \right|^2 dx$$

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Hence it is not viscosity

but viscosity-capillarity

entropy equality as one

would see from Korteweg's

theory of capillarity

$C - E \Rightarrow$ non-local

visco-capillarity entropy

equality.

Who wins the competition? 28

Viscosity or capillarity?

$$\frac{3}{5} \hat{P}_t = \nu \hat{u}$$

$$\hat{u}_t = \nu (\hat{p} - \nu A(r) \hat{u} - r^2 B(r^2) \hat{p})$$

$$\frac{3}{5} \hat{P}_{tt} + r^2 \hat{p} + r^2 (-A(r) \frac{3}{5} \hat{P}_t - r^2 B(r^2) \hat{p}) = 0$$

Try to write as

$$\left(\frac{3}{5} \frac{d}{dt} + \sigma_1(r) \right) \left(\frac{d}{dr} + \sigma_2(r) \right) \hat{p} = -r^2 \hat{p}$$

$$\Rightarrow \left\{ \begin{array}{l} \sigma_1(r) + \frac{3}{5} \sigma_2(r) = -\frac{3}{5} r^2 A(r^2) \\ \sigma_1(r) \sigma_2(r) = -r^4 B(r^2) \end{array} \right\}$$

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$$\frac{d\hat{p}}{dt} + \sigma_2(k) \hat{p} = ik\hat{v}$$

$$\Rightarrow \left(\frac{3}{5} \frac{d}{dt} + \sigma_F(k) \right) (ik\hat{v}) = -n^2 p$$

$$\frac{3}{5} \frac{d}{dt} \hat{v} = ik\hat{p} - \sigma_1(k) \hat{v}$$

$$\frac{d\hat{p}}{dt} = ik\hat{v} - \sigma_2(k) \hat{p}$$

$$\frac{3}{5} \partial_t v = -\partial_x p - \frac{1}{12\pi} \int_{-\infty}^{\infty} e^{ikx} \sigma_1(k) \hat{v}(k, t) dk$$

$$\partial_t p = -\partial_x v - \frac{1}{12\pi} \int_{-\infty}^{\infty} e^{ikx} \sigma_2(k) \hat{p}(k, t) dk$$

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$$\nabla_1(r) = -\frac{3}{5} (\nabla_2(r) + r^2 A(r^2))$$

\Rightarrow

$$-\frac{3}{5} \nabla_2(r) (\nabla_2(r) + r^2 A(r^2)) = -r^4 B(r^2)$$

$$-\frac{3}{5} \nabla_2(r)^2 - \frac{3}{5} r^2 \nabla_2(r) A(r^2) + r^4 B(r^2) = 0$$

$$\nabla_2(r)^2 + r^2 \nabla_2(r) A(r^2) - \frac{5}{3} r^4 B(r^2) = 0$$

$$2 \nabla_2(r) = -r^2 A(r^2)$$

$$\pm \left((r^2 A(r^2))^2 + \frac{4}{3} \left(+5 \right) R^4 B(r^2) \right)^{\frac{1}{2}}$$

$$\begin{aligned} 2 \nabla_2(r) &= -\left(\frac{C}{1-C}\right) \\ &\pm \left(\left(\frac{C}{1-C}\right)^2 + \frac{20}{3} r^2 C \right)^{\frac{1}{2}} \end{aligned}$$

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$$ZG_2(r) = - \left(\frac{C}{1-C} \right)$$

$$\pm \left(-\frac{5}{3} r^2 C \left(C + \frac{4}{5} \right) + \frac{20}{3} r^2 C \right)^{1/2}$$

$$= - \left(\frac{C}{1-C} \right) \pm |r| \left(-\frac{5}{3} C \left(C + \frac{4}{5} \right) + \frac{20}{3} C \right)^{1/2}$$

$$= - \left(\frac{C}{1-C} \right) \pm c |r| \left(\frac{5}{3} C^2 + \frac{5}{3} \frac{4}{5} C - \frac{20}{3} C \right)^{1/2}$$

$$= - \left(\frac{C}{1-C} \right) \pm c |r| \left(\frac{5}{3} C^2 - \frac{16}{3} C \right)^{1/2}$$

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\Rightarrow

$$2\sigma_1(n) + \frac{3}{5} (2\sigma_2(n) + 2\frac{c}{1-c}) = 0$$

$$2\sigma_1(n) + \frac{3}{5} \left(-\left(\frac{c}{1-c}\right) + \left(\frac{2c}{1-c}\right) \right) \\ \pm 1_{|k|} \left(5c^2 - 16c \right)^{1/2}$$

$$2\sigma_1(n) + \frac{3}{5} \left(\left(\frac{c}{1-c}\right) \pm 1_{|k|} \left(5c^2 - 16c \right)^{1/2} \right) = 0$$

$$\boxed{2\sigma_1(n) = -\frac{3}{5} \left(\frac{c}{1-c}\right)} \\ \boxed{\pm \frac{3}{5} 1_{|k|} \left(5c^2 - 16c \right)^{1/2}}$$

Capillarity wins.